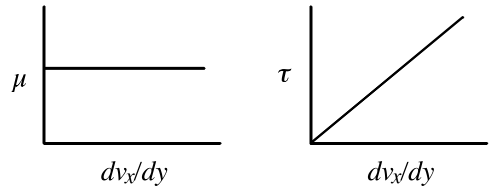
I. Non-Newtonian Fluids - Overview

A. Simple Classifications of Fluids

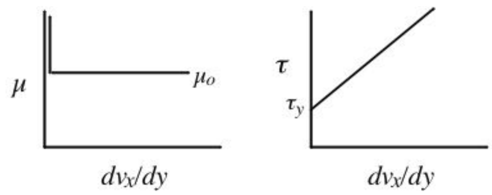
1. Shear-Dependent Fluids

a. Newtonian
(obeys Newton's
Law of viscosity)



$$\tau = \mu \frac{dv_x}{dy}$$

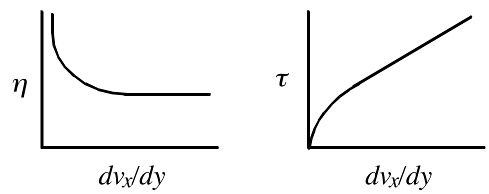
b. Bingham Plastics
(ideal plastic)



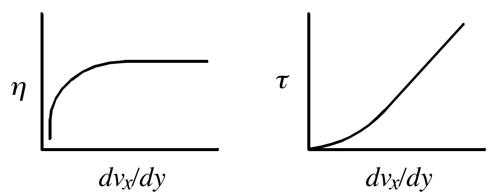
$$\tau = \mu_o \frac{dv_x}{dy} + \tau_y \quad for\ |\tau| > \tau_y$$

$$\frac{dv_x}{dy} = 0\ for\ |\tau| \le \tau_y$$

c. Pseudoplastic
(molecules line up
once flow begins)



d. Dilatant
(particle suspensions
collide when flow
begins)



Can be fit with the Power Law

$$\tau = \kappa \left| \frac{dv_x}{dy} \right|^n$$

OR with other models

Reminder: Graduate texts write Newton's Law as $\tau = -\mu\, dv_x/dy$

e. More complex models

i) Powell-Eyring Model (based on Eyring's theories of the molecular "structure" of liquids)

$$\tau = A\frac{dv_x}{dy} + \frac{1}{B}\sinh^{-1}\left(\frac{1}{C} \bullet \frac{dv_x}{dy}\right)$$

ii) Ellis Model (purely empirical)

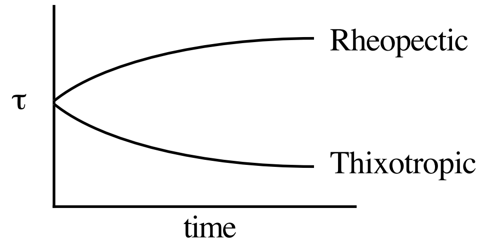
$$\tau = \frac{1}{A + B\tau^C} \bullet \frac{dv_x}{dy}$$

iii) Reiner-Phillipoff Model (purely empirical)

$$\tau = \left(A + \frac{B - A}{1 + (\tau/C)^2}\right)\frac{dv_x}{dy}$$

iv) For each of these, the three arbitrary constants (A, B, and C) are adjusted to fit the data.

2. Time-Dependent Fluids

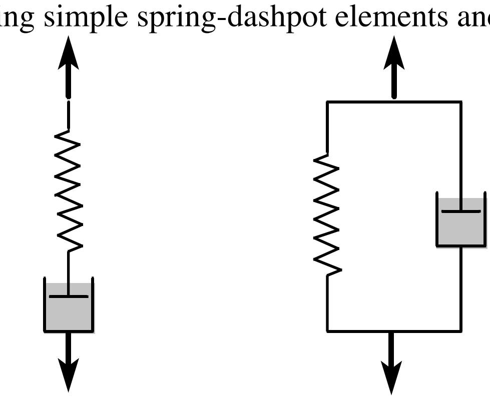


motion causes aggregation or unraveling of molecules

motion causes re-coiling or fragmenting of molecules

   a. The changes occur mostly within the first 60 seconds
   b. Description of these fluids is extremely difficult.

3. Viscoelastic Fluids
   a. Exhibit elastic "recovery" from deformations which occur during flow
   b. Also exhibit "plastic" (permanent) deformation
   c. Modeled using simple spring-dashpot elements and combinations of those elements



Maxwell Element   Kelvin or Voigt Element

B. Observations
1. Non-Newtonian behavior is usually seen when the fluid is composed of small carrier molecules and larger suspended molecules or particles.
   a. Bingham Fluids: Coal slurries, grain slurries, sewage sludge
   b. Pseudoplastic Fluids: Polymer melts, paper pulp suspensions, pigment suspensions
   c. Dilatant Fluids: Starch suspensions, mica suspensions, quicksand
   d. Thixotropic Fluids: Mayonnaise, drilling mud, paint, ink
   e. Rheopectic Fluids: Gypsum suspensions in water, vanadium pentoxide sols
   f. Viscoelastic Fluids: Polymeric liquids
   g. Other: Yield Pseudoplastic Fluid: Blood
2. Non-Newtonian fluids exhibit laminar and turbulent flow regimes, with a transition between those regimes.
3. Equations for Newtonian fluids can't be used, because there is no single value of "viscosity" for such fluids.

   e.g. Can't calculate the Reynolds number using the Newtonian definition

4. We often talk about an "apparent viscosity" (η) where

$$\tau = \eta \frac{dv_x}{dy} \qquad (1)$$

   a. For example for a power-law fluid

$$\tau = K\left(\frac{dv_x}{dy}\right)^n \qquad (2)$$

   so, for such a fluid

$$\eta = K\left(\frac{dv_x}{dy}\right)^{n-1} \qquad (3)$$

   b. The apparent viscosity varies with shear rate, as plotted on page 1.

II. Describing Non-Newtonian <u>Laminar</u> Flow in <u>Horizontal Cylindrical Pipes</u>

A. Describing the Wall Shear Stress (all fluids)

1. As in Class 16, we can begin with the Equation of Motion to derive

$$-\tau_{rz} = \frac{1}{2}\left(\frac{P_1 - P_2}{L} + \rho g \sin\alpha\right) r \qquad (4)$$

2. For horizontal pipes

$$-\tau_{rz} = \frac{r}{2}\frac{P_1 - P_2}{L} = -\frac{r}{2}\frac{dP}{dz} \qquad (5)$$

3. At the wall

$$\tau_w = \frac{D}{4}\frac{dP}{dz} \qquad (6)$$

(which has a negative value, because it acts in the $-z$ direction)

B. Velocity Profile and Flowrate

1. For laminar flow, we could introduce the rheological equation into The Equation of Motion (or into Equation (5) above) and integrate to get $v_z(r)$.

Example: For a Power-Law fluid, Çengel's convention for shear stress gives

$$-\tau_{rz} = K\left|\frac{dv_z}{dr}\right|^n = K\left(-\frac{dv_z}{dr}\right)^n$$

•Combining with Equation (5), integrating, and applying the no-slip condition gives

$$v_z = \left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{1}{n}} \frac{n}{n+1}\left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right) \qquad (7)$$

•The volumetric flow rate becomes

$$Q = \left(\frac{-dP}{dz}\frac{D}{4K}\right)^{\frac{1}{n}} \frac{\pi n D^3}{8(3n+1)} \qquad (8)$$

•And, finally (see the Appendices to these notes),

$$\beta = \frac{3n+1}{2n+1} \qquad \text{and} \qquad \alpha = \frac{3(3n+1)^2}{(5n+3)(2n+1)} \tag{9}$$

2. For other types of non-Newtonian fluids, the math often becomes very complex.

C. How can we apply these concepts to typical pipeflow data ($-dP/dz$ vs. v_{avg})?
A General Approach, Using Newtonian Fluids in Laminar Flow as a Model

1. The velocity profile for a Newtonian fluid is

$$v_z = v_{\max}\left(1-\frac{r^2}{R^2}\right) = 2v_{avg}\left(1-\frac{r^2}{R^2}\right) \tag{10}$$

2. So the shear rate is

$$\dot{\gamma} = -\frac{dv_z}{dr} = -2v_{avg}\left(-\frac{2r}{R^2}\right) = \frac{4v_{avg}r}{R^2} \tag{11}$$

3. At the wall

$$\dot{\gamma}_w = \frac{4v_{avg}}{R} = \frac{8v_{avg}}{D} \tag{12}$$

4. For a Newtonian fluid, a short-hand way to write Newton's Law is

$$\tau_w = \mu\dot{\gamma}_w \tag{13}$$

so plotting τ_w vs. $\frac{8v_{avg}}{D}$ gives a straight line of slope = μ

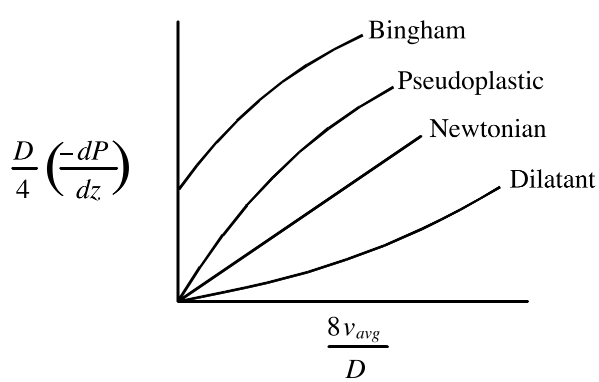
5. But

$\tau_w = \frac{D}{4}\frac{dP}{dz}$ (from Equation 6) and

$\dot{\gamma}_w = \frac{8v_{avg}}{D}$ (from Equation 12)

So plotting $-\frac{D}{4}\frac{dP}{dz}$ vs $\frac{8v_{avg}}{D}$ for various diameters gives a straight line of slope = μ

6. Non-Newtonian fluids can be plotted on that same plot (gives a generalized way of representing all fluids in laminar flow).



D. How do we design flow systems for a Power-Law fluid in laminar flow?

1. From Equation 8,

$$\dot{V} = \left(\frac{-dP}{dz}\frac{D}{4K}\right)^{\frac{1}{n}} \frac{\pi n D^3}{8(3n+1)}$$

Solving for pressure drop:

$$\frac{-dP}{dz} = \left[\dot{V}\frac{8(3n+1)}{\pi n D^3}\right]^n \frac{4K}{D} \approx -\frac{P_2 - P_1}{L}$$

But the mechanical energy equation uses $\Delta P/\rho$, so

$$\frac{P_2 - P_1}{\rho} = \left(\frac{P_2 - P_1}{L}\right)\frac{L}{\rho} = -\left[\dot{V}\frac{8(3n+1)}{\pi n D^3}\right]^n \frac{4KL}{D\rho}$$

2. Finally, from the mechanical energy equation for a horizontal pipe,

$$\frac{P_2 - P_1}{\rho} = -w_f$$

3. So for laminar pipe flow of a power-law fluid,

$$\boxed{w_f = \left[\dot{V}\frac{8(3n+1)}{\pi n D^3}\right]^n \frac{4KL}{D\rho}}$$

which can be used with the mechanical energy equation for any applicable piping system.

Appendix 1 – β for a Power-Law Fluid in Laminar Pipe Flow

$$Q \;=\; \iint v\, dA \;=\; \left(-\frac{dP}{dz}\frac{D}{4K}\right)^{\frac{1}{n}} \frac{\pi n R^3}{3n+1} \qquad \text{(see Equation 8 of these notes)}$$

$$\iint v^2\, dA \;=\; \iint \left[\left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{1}{n}} \frac{n}{n+1}\left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)\right]^2 r\, dr\, d\theta \qquad \text{(see Equation 7 of these notes)}$$

$$= \; 2\pi \left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{2}{n}} \left(\frac{n}{n+1}\right)^2 \int_0^R \left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right) r\, dr$$

where

$$\int_0^R \left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)^2 r\, dr \;=\; \int_0^R \left(R^{2+\frac{2}{n}} - 2R^{1+\frac{1}{n}} r^{1+\frac{1}{n}} + r^{2+\frac{2}{n}}\right) r\, dr$$

$$= \left[R^{2+\frac{2}{n}} \frac{r^2}{2} - 2R^{1+\frac{1}{n}} \frac{r^{3+\frac{1}{n}}}{3+\frac{1}{n}} + \frac{r^{4+\frac{2}{n}}}{4+\frac{2}{n}}\right]_0^R$$

$$= \; \frac{R^{4+\frac{2}{n}}}{2} + \frac{2R^{4+\frac{2}{n}}}{3+\frac{1}{n}} + \frac{R^{4+\frac{2}{n}}}{4+\frac{2}{n}} \;=\; R^{4+\frac{2}{n}}\left[\frac{1}{2} - \frac{2n}{3n+1} + \frac{n}{4n+2}\right]$$

which, after some math, gives

$$= \; R^{4+\frac{2}{n}} \frac{(n+1)^2}{(3n+1)(4n+2)}$$

So $\displaystyle \iint v^2\, dA \;=\; 2\pi \left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{2}{n}} \left(\frac{n}{n+1}\right)^2 R^{4+\frac{2}{n}} \frac{(n+1)^2}{(3n+1)(4n+2)}$

$$= \; \pi R^4 \left(\frac{-dP}{dz}\frac{D}{4K}\right)^{\frac{2}{n}} \frac{n^2}{(3n+1)(2n+1)}$$

Finally, $\displaystyle \beta \;=\; A\frac{\iint v^2\, dA}{\left[\iint v\, dA\right]^2} \;=\; \pi R^2 \frac{\pi R^4 \left(\frac{-dP}{dz}\frac{D}{4K}\right)^{\frac{2}{n}} \frac{n^2}{(3n+1)(2n+1)}}{\left(-\frac{dP}{dz}\frac{D}{4K}\right)^{\frac{2}{n}} \frac{\pi^2 n^2 R^6}{(3n+1)^2}} \;=\; \frac{3n+1}{2n+1}$

Appendix 2 – α for a Power-Law Fluid in Laminar Pipe Flow

$$Q = \iint v\, dA = \left(-\frac{dP}{dz}\frac{D}{4K}\right)^{\frac{1}{n}} \frac{\pi n R^3}{3n+1} \qquad \text{(see Equation 8 of these notes)}$$

$$\iint v^3\, dA = \iint \left[\left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{1}{n}} \frac{n}{n+1}\left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)\right]^3 r\, dr\, d\theta \qquad \text{(see Equation 7 of these notes)}$$

$$= 2\pi\left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{3}{n}}\left(\frac{n}{n+1}\right)^3 \int_0^R \left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)^3 r\, dr$$

where

$$\int_0^R \left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)^3 r\, dr = \int_0^R \left(R^{3+\frac{3}{n}} - 3R^{2+\frac{2}{n}} r^{1+\frac{1}{n}} + 3R^{1+\frac{1}{n}} r^{2+\frac{2}{n}} - r^{3+\frac{3}{n}}\right) r\, dr$$

$$= \left[R^{3+\frac{3}{n}}\frac{r^2}{2} - 3R^{2+\frac{2}{n}}\frac{r^{3+\frac{1}{n}}}{3+\frac{1}{n}} + 3R^{1+\frac{1}{n}}\frac{r^{4+\frac{2}{n}}}{4+\frac{2}{n}} - \frac{r^{5+\frac{3}{n}}}{5+\frac{3}{n}}\right]_0^R$$

$$= \frac{R^{5+\frac{3}{n}}}{2} - \frac{3R^{5+\frac{3}{n}}}{3+\frac{1}{n}} + \frac{3R^{5+\frac{3}{n}}}{4+\frac{2}{n}} - \frac{R^{5+\frac{3}{n}}}{5+\frac{3}{n}}$$

$$= R^{5+\frac{3}{n}}\left[\frac{1}{2} - \frac{3n}{3n+1} + \frac{3n}{4n+2} - \frac{n}{5n+3}\right]$$

which, after some math, gives

$$= R^{5+\frac{3}{n}} \frac{3(n+1)^3}{(3n+1)(4n+2)(5n+3)}$$

So

$$\iint v^3\, dA = 2\pi\left(\frac{-dP}{dz}\frac{1}{2K}\right)^{\frac{3}{n}}\left(\frac{n}{n+1}\right)^3 R^{5+\frac{3}{n}} \frac{3(n+1)^3}{(3n+1)(4n+2)(5n+3))}$$

$$= \pi R^5 \left(\frac{-dP}{dz}\frac{D}{4K}\right)^{\frac{3}{n}} \frac{3n^3}{(3n+1)(2n+1)(5n+3)}$$

Finally,

$$\alpha = A^2 \frac{\iint v^3\, dA}{\left[\iint v\, dA\right]^3} = \pi^2 R^4 \frac{\pi R^5 \left(\frac{-dP}{dz}\frac{D}{4K}\right)^{\frac{3}{n}} \frac{3n^3}{(3n+1)(2n+1)(5n+3)}}{\left(-\frac{dP}{dz}\frac{D}{4K}\right)^{\frac{3}{n}} \frac{\pi^3 n^3 R^9}{(3n+1)^3}} = \frac{3(3n+1)^2}{(2n+1)(5n+3)}$$

I. More On Non-Newtonian Laminar Flow in Horizontal Cylindrical Pipes

A. Determining Values of the Rheological Parameters

1. **We can measure flow rate and pressure drop through a pipeline... So how do we determine the values of rheological parameters?**
2. We can show (appendix for these notes) that, for all non-Newtonian fluids

$$\dot{\gamma}_w = \xi\left[\frac{3}{4} + \frac{1}{4}\frac{d\ln\xi}{d\ln\tau_w}\right] \tag{1}$$

where $\dot{\gamma}_w = \left(-\frac{dv_z}{dr}\right)_w$, $\xi = \frac{8v_{avg}}{D}$, and $\tau_w = \frac{D}{4}\left(-\frac{dP}{dz}\right)$

3. Equation 1, called the Rabinowitsch-Mooney equation, works for most time-independent non-Newtonian fluids in laminar flow in cylindrical pipes.
   a. From data of flow rate and pressure drop, ξ and τ_w can be calculated for each condition (data point)
   b. A plot of ln τ_w vs. ln ξ will produce
   $$\frac{d\ln\tau_w}{d\ln\xi}$$
   which can be inserted into the right-hand bracket of Equation 1.
   c. The wall shear rate can then be determined for each data point.
   d. Equation 1 suggests that $\frac{d\ln\tau_w}{d\ln\xi}$ provides important information.

4. The values of rheologic parameters can also be found from plots involving τ_w and ξ.
   a. For Newtonian fluids, $\frac{d\ln\tau_w}{d\ln\xi} = 1$.
      i) Why is this so?
      Hint: How does τ_w depend on ξ? To find out...
      How does τ_w depend on $\dot{\gamma}$?
      How does $\dot{\gamma}$ depend on ξ?
      $$\dot{\gamma}_w = -\frac{dv_z}{dr} = \frac{8v_{avg}}{D} = \xi$$
      (Now, you finish the story.)

      ii) So, what kind of plot of the data would tell us μ?

b. For <u>Power-Law fluids</u>, $\frac{d\ln\tau_w}{d\ln\xi} = n$

i) Why is this so?

Hint: How does τ_w depend on ξ? To find out...

How does τ_w depend on $\dot{\gamma}$?

How does $\dot{\gamma}$ depend on ξ?

$$\tau_w = K\left(-\frac{dv_z}{dr}\right)_w^n = K\dot{\gamma}_w^n$$

$$\dot{\gamma}_w = \left[\frac{\tau_w}{K}\right]^{1/n} = \left[-\frac{dP}{dz}\frac{D}{4K}\right]^{1/n}$$

$$\dot{V} = \left[-\frac{dP}{dz}\frac{D}{4K}\right]^{1/n}\frac{\pi n D^3}{8(3n+1)} = \dot{\gamma}_w\frac{\pi n D^3}{8(3n+1)}$$

$$\dot{\gamma}_w = \frac{8(3n+1)\dot{V}}{\pi n D^3} = \frac{4\dot{V}}{\pi D^2}\frac{8}{D}\frac{3n+1}{4n} = \frac{8v_{avg}}{D}\frac{3n+1}{4n} = \xi\frac{3n+1}{4n}$$

(Now, you finish the story.)

ii) So, what kind of plot of the data would tell us n?

iii) Would all data sets (all pipe diameters, etc.) give the same value of n?

iv) Once we know the value of n, can we determine the value of K? How?

v) What are the units of K?

c. For other Non-Newtonians fluids, $\frac{d\ln\tau_w}{d\ln\xi}$ may be a simple function, and we may be able to do the same kind of analysis to determine values of rheologic parameters.

B. When Does Flow Become Non-Laminar?

1. For Newtonian fluids: when Reynolds Number ($Dv\rho/\mu$) > ~2100
2. But, for non-Newtonian fluids, there is no "μ".
3. For horizontal, cylindrical pipes,

$$\frac{\Delta P}{\rho} = -f\frac{L}{D}\frac{v^2}{2} \tag{2}$$

Rearranging

$$f = -\frac{D}{4}\frac{\Delta P}{L}\frac{8}{\rho v^2} = \tau_w\frac{8}{\rho v^2} \tag{3}$$

4. For Power-Law Fluids

a. By definition

$$\tau_w = K(\dot{\gamma}_w)^n = K\left(\frac{8v_{avg}}{D}\frac{3\text{n}+1}{4\text{n}}\right)^{\text{n}} \approx K\left(\frac{8v_{avg}}{D}\right)^n \quad \text{(approximate derivation)} \tag{4}$$

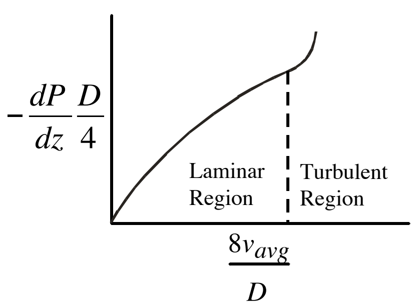
b. We can substitute into Equation 3 to obtain

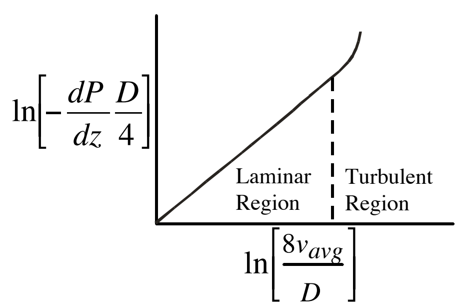
$$f \approx K\left(\frac{8v}{D}\right)^n\frac{8}{\rho v^2} \tag{5}$$

c. Finally, if we assume that for laminar flow, f≈64/Re (from Newtonian fluid)

$$\text{Re} \approx \frac{64}{K\left(\frac{8v}{D}\right)^n\frac{8}{\rho v^2}} \approx \frac{D^n\rho v^{2-n}}{K8^{n-1}} \tag{6}$$

d. It turns out that when this "Generalized" Reynolds number (for Power-Law Fluids) exceeds ~2100, the flow of Power-Law fluids departs from laminar behavior.





e. Caution: The location of the turbulent break-off point in the above graph varies with pipe diameter.

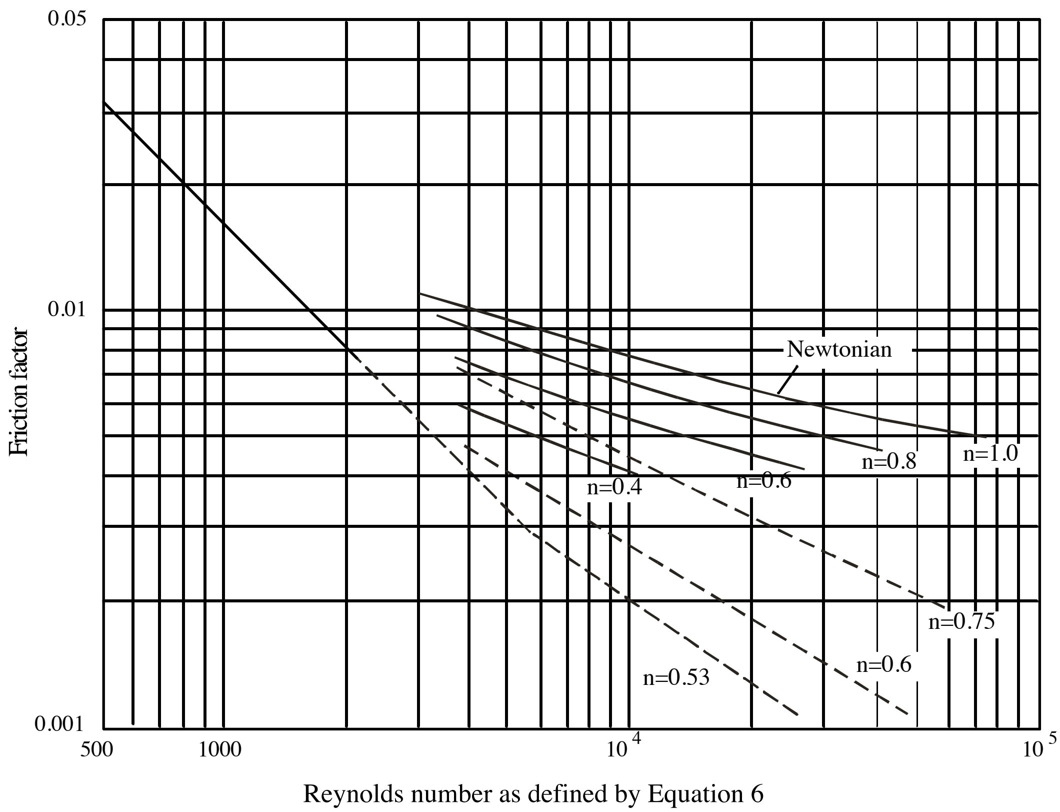
5. For other kinds of fluids, Equations 4-6 can be re-derived for the particular fluid in question, resulting in a special formulation of the Reynolds number for that fluid.

II. Describing Non-Newtonian Turbulent Flow in Horizontal Cylindrical Pipes

A. Empirical Friction-Factor Charts Are Used

The following gives $f_{Fanning} = f_{Darcy}/4$ for Power-Law Fluids.

(From N. de Nevers, Fluid Mechanics, Addison Wesley, 1970.)



B. The solid lines in the above graph are for slurries and some polymer solutions.
(From the data of Dodge and Metzner, *AIChE J*. **5**:189, 1959.)

C. The dotted lines are for some polymer solutions and polymer melts, particularly those which show viscoelastic behavior (like rubber cement), which suppress turbulence
(From the data of Shaver and Merrill, *AIChE J*. **5**:181, 1959.)

1. With these fluids, much less turbulence is seen, and friction is reduced.

2. Apparently, the long molecules of these fluids absorb energy and damp out turbulent fluctuations.

3. Tiny amounts (as little as 0.01 wt%) of such polymers in water reduce friction 50%.

4. Such pressure-loss-reduction additives are commonly used in the industry (e.g. the petroleum industry).

D. The friction factor from the above graph is used in a similar way as for Newtonian fluid ($w_f = 4f\, L/D\, v^2/2$, used in the mechanical energy equation as always).

Appendix - Derivation of the Rabinowitsch-Mooney Equation

1. From the definition of the volumetric flow rate

$$\dot{V} = \int_0^R \int_0^{2\pi} v\, r\, d\theta\, dr = 2\pi \int_0^R v\, r\, dr \tag{A1}$$

2. But $\int x\, dy = xy - \int y\, dx$, and if we let $dy = r\, dr$ (so $y = \frac{1}{2} r^2$), then

$$\dot{V} = 2\pi \left[\frac{vr^2}{2} \bigg|_{r=0}^{r=R} - \frac{1}{2} \int_{r=0}^{r=R} r^2 dv \right] = -\pi \int_{r=0}^{r=R} r^2 dv \tag{A2}$$

3. We can write this as

$$\dot{V} = -\pi \int_0^R r^2 \left(\frac{dv}{dr} \right) dr = \pi \int_0^R r^2 \dot{\gamma}\, dr \tag{A3}$$

4. From the last class,

$$\tau_{rz} = -\frac{r}{2}\frac{dP}{dz} \qquad \text{and} \qquad \tau_w = -\frac{D}{4}\frac{dP}{dz}$$

so

$$r = \frac{D}{2}\frac{\tau}{\tau_w} \qquad \text{and} \qquad dr = \frac{D}{2\tau_w} d\tau \tag{A4}$$

5. Substituting into Equation A3,

$$\dot{V} = \frac{\pi D^3}{8\tau_w^3} \int_0^{\tau_w} \dot{\gamma}\, \tau^2 d\tau \tag{A5}$$

6. Clearing the pre-integral fraction on the right side

$$\int_0^{\tau_w} \dot{\gamma}\, \tau^2 d\tau = \frac{8\dot{V}}{\pi D^3} \tau_w^3 = \frac{1}{4} \frac{8 v_{avg}}{D} \tau_w^3 = \frac{1}{4} \xi \tau_w^3 \tag{A6}$$

where we have defined $\xi \equiv \dfrac{8 v_{avg}}{D}$

7. Noting that $d\left[\int_a^b F'(x) dx \right] = d\left[F(b) - F(a) \right] = F'(b)\, db - F'(a)\, da$,

we take the derivatives of both sides of equation (A6) to obtain

$$\frac{1}{4} \xi (3\tau_w^2\, d\tau_w) + \frac{1}{4} \tau_w^3\, d\xi = \dot{\gamma}_w \tau_w^2\, d\tau_w \tag{A7}$$

8. Finally, rearranging

$$\dot{\gamma}_w = \xi \left[\frac{3}{4} + \frac{1}{4} \frac{d\xi}{\xi} \frac{\tau_w}{d\tau_w} \right] \tag{A8}$$

or

$$\dot{\gamma}_w = \xi \left[\frac{3}{4} + \frac{1}{4} \frac{d \ln \xi}{d \ln \tau_w} \right] \tag{A9}$$

where $\dot{\gamma}_w = \left(-\dfrac{dv_z}{dr} \right)_w$, $\xi = \dfrac{8 v_{avg}}{D}$, and $\tau_w = \dfrac{D}{4} \left(-\dfrac{dP}{dz} \right)$